Improved Time-Stepping for the BISICLES Ice-Sheet Model

FASTMath Seminar

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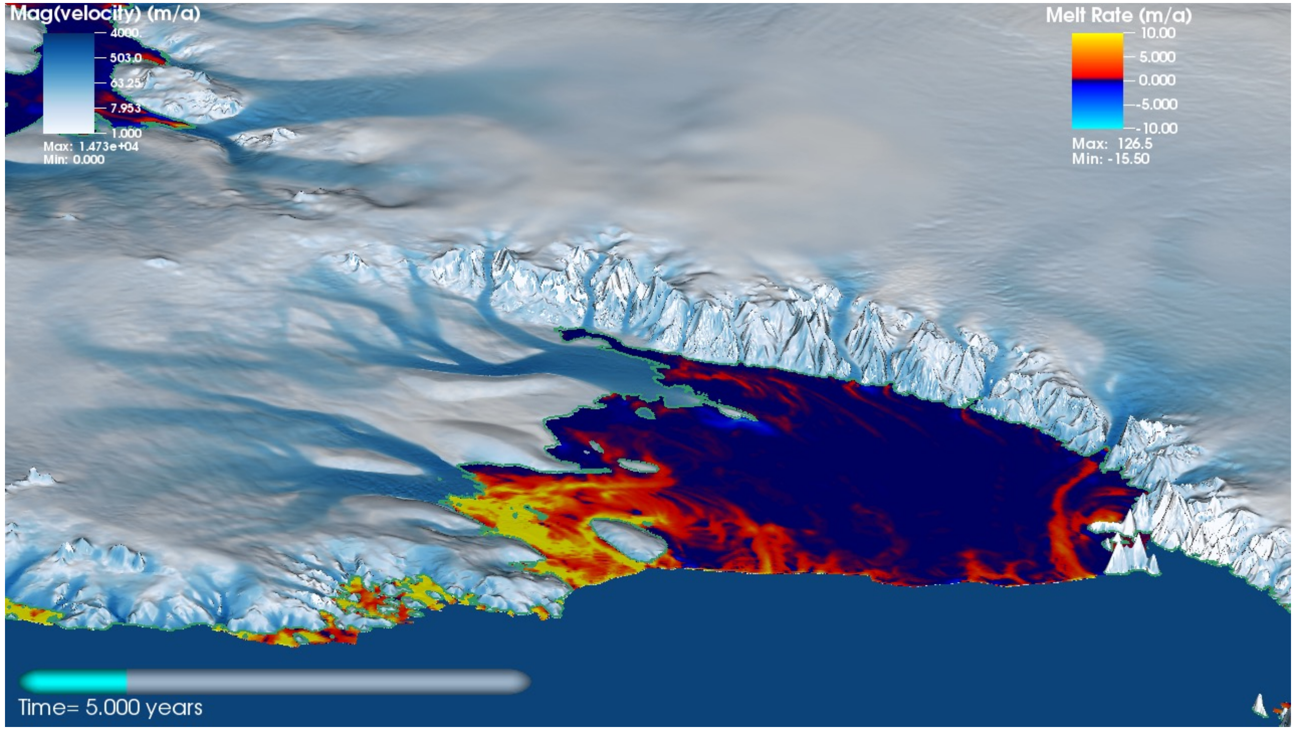
BISICLES Models Ice Sheet Dynamics

- Accurate modeling of ice-sheets is critical to understanding and predicting
  - Future sea level rise
  - Potential regional collapses in the West Antarctic ice sheet
- BISICLES is a simulation tool developed at LBNL, LANL, and the University of Bristol[1]
- Long-term time evolution of these models requires accurate, conservative, and stable numerical methods



1. Cornford, Stephen L., et al. "Adaptive mesh, finite volume modeling of marine ice sheets." *JCP* 232.1 (2013): 529-549.

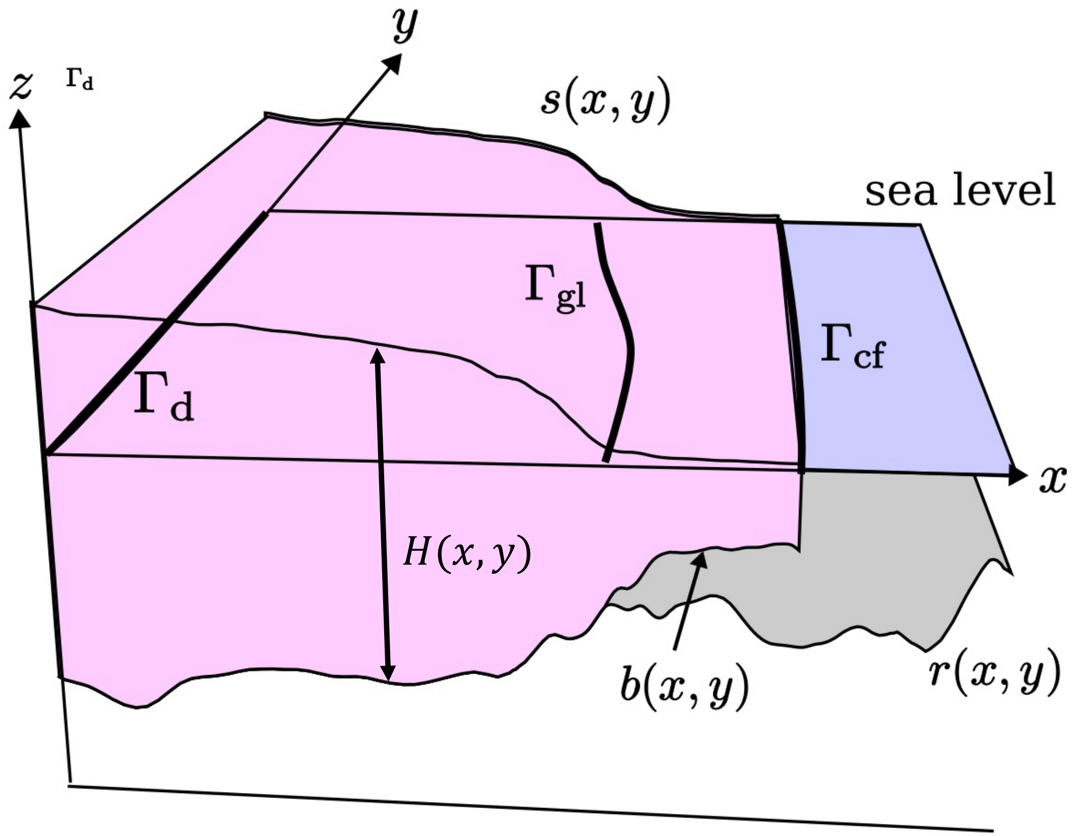
The Ice Model Combines Hyperbolic and Elliptic Partial Differential Equations

- In the simplest case, the two primary variables are
  - Ice thickness $H(t, x, y)$
  - Ice velocity $v(t, x, y)$
- An asymptotically-derived approximation to Stokes Flow is used[1]

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}(v_x H) + \frac{\partial}{\partial y}(v_y H)$$

$$\beta^2 v - \nabla \cdot (H\mu(v)\, \nabla v) = -\rho_i g H\, \nabla \cdot s$$

- The Chombo library[2] is used for the spatial discretization with adaptive mesh refinement (AMR)
  - Second order finite volume method

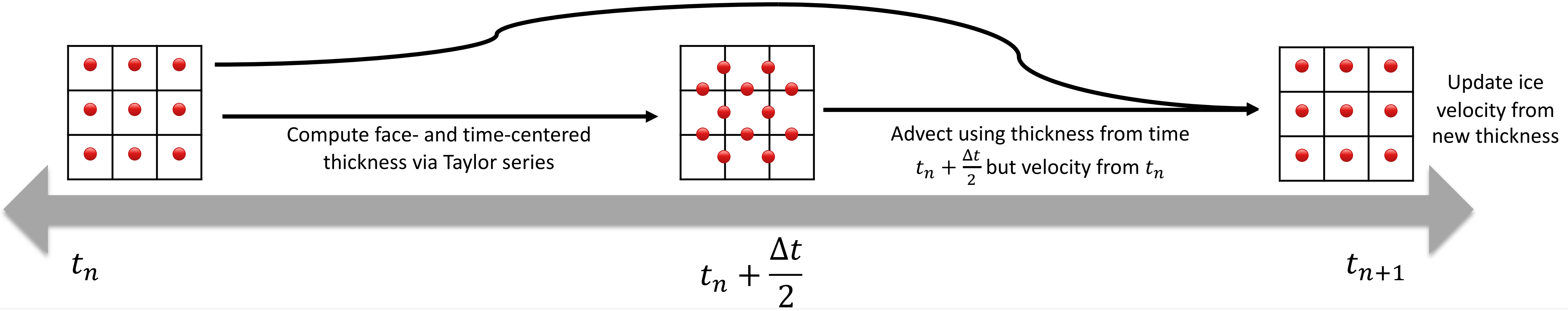


From Cornford, Stephen L., et al. "Adaptive mesh, finite volume modeling of marine ice sheets." *JCP* 232.1 (2013): 529-549.

1. Schoof, Christian, and Richard CA Hindmarsh. "Thin-film flows with wall slip: an asymptotic analysis of higher order glacier flow models." *QJMAM* 63.1 (2010): 73-114.
2. Colella, Phillip, et al. "Chombo software package for AMR applications design document." (2009).

BISICLES was Limited by the Time Discretization

- BISICLES uses an unsplit Godunov piecewise parabolic method
  - First order accurate in time
  - Explicit
  - Not method of lines
  - Limited maximum stable time step
  - No error estimation
  - Time step chosen by CFL condition based on assumption
- Project Goals
  - Introduce high order time-stepping methods for improved accuracy and stability
    - The focus will be explicit methods
    - Implicit methods are sensible but difficult to implement
  - Introduce adaptive timestep methods
  - Determine which class of integrators is best-suited to the problem



We can Solve an Ordinary or Differential-Algebraic Equation

- The time evolution problem is an index-1 differential-algebraic equation (DAE)

$$\frac{dH}{dt} = f(H, v)$$
$$0 = g(H, v)$$

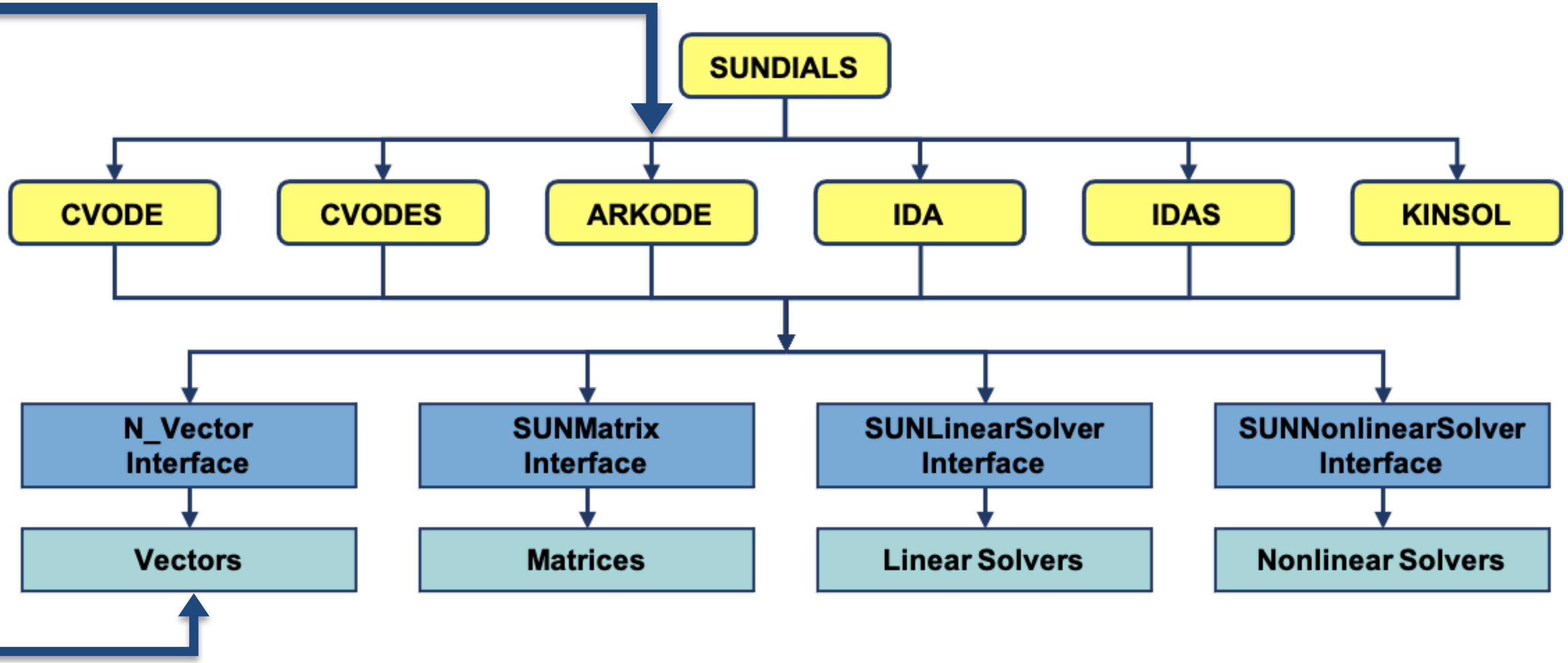
- Over 90% of the runtime is spent solving the nonlinear system $0 = g(H, v)$!
- Or we can view this as an ordinary differential equation (ODE) where $v = \mathcal{G}(H)$ is a derived quantity computed via a nonlinear solve. This is the "state space form"[1]

$$\frac{dH}{dt} = f\left(H, \mathcal{G}(H)\right)$$

1. Wanner, Gerhard, and Ernst Hairer. *Solving ordinary differential equations II*. Vol. 375. New York: Springer Berlin Heidelberg, 1996. Section VI.1

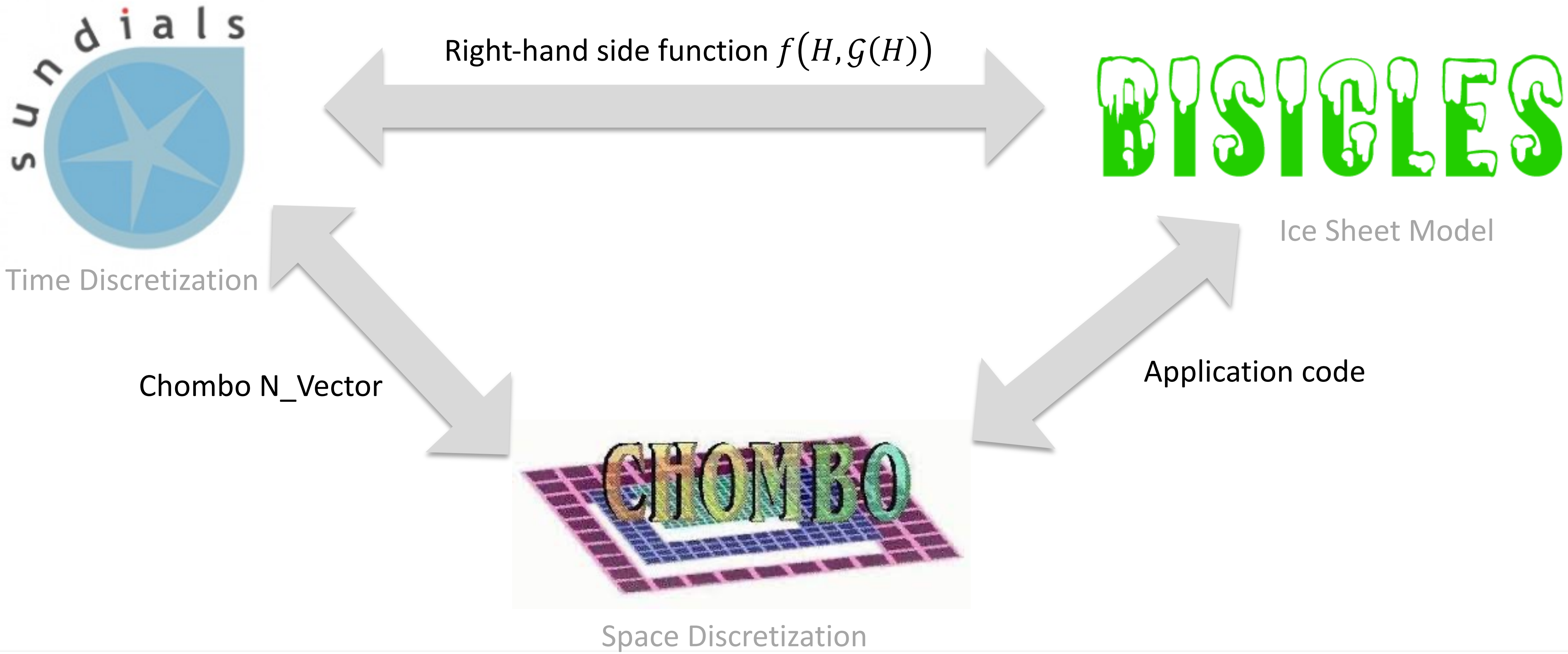
SUNDIALS Provides Efficient ODE, DAE, and Nonlinear Solvers

- ARKODE provides (additive) Runge-Kutta methods
- Adaptive or fixed step size
- We use explicit Runge-Kutta methods to solve the state space form $\frac{dH}{dt} = f\big(H, \mathcal{G}(H)\big)$



- N_Vectors decouple integrators from application data structures
- Includes norms, dot product, axpy, and other generic operations
- **We developed a Chombo N_Vector to operate on AMR grids**

Our New Chombo N_Vector Enables Package Interoperability



Second Order is Feasible at the Cost of Order One

$$\frac{dH}{dt} = f(H, v)$$
$$0 = g(H, v)$$

- BISICLES' first order integrator does **1** expensive algebraic solve for ice velocity each time step
- A second order, explicit Runge-Kutta applied to the state space form requires at least **2** algebraic solves per time step
- We proposed to use a second order "half-explicit[1] Heun's method" with **1** algebraic solve per step

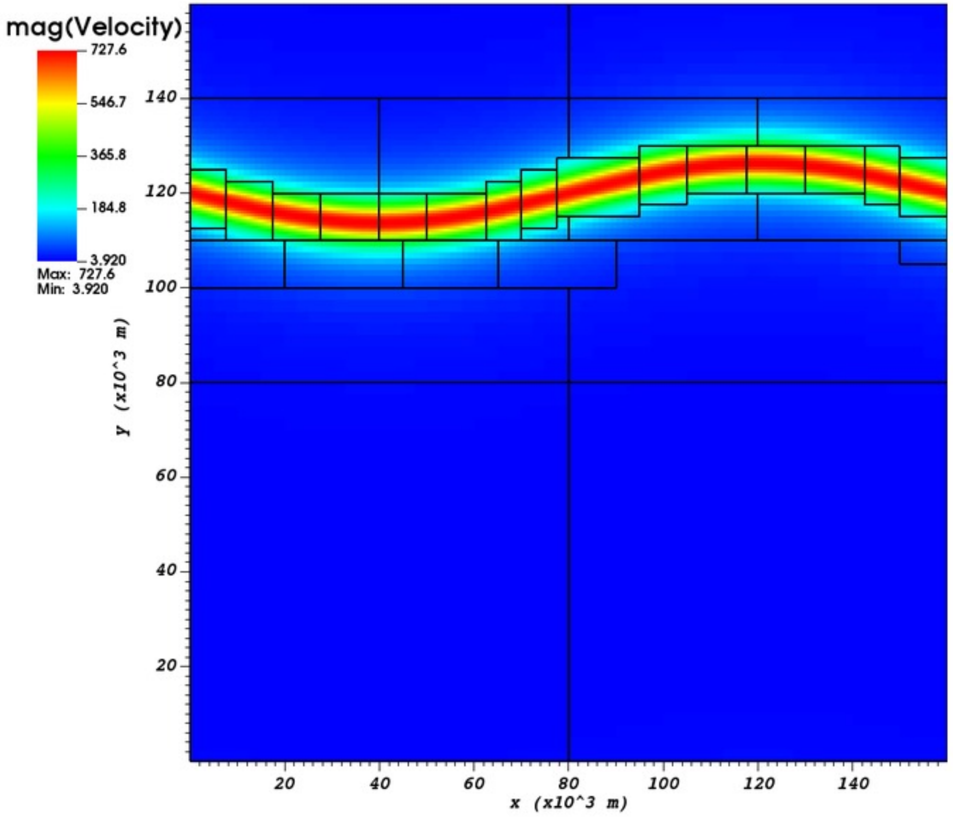
$$K_1 = f(H_n, v_n)$$
$$K_2 = f(H_n + \Delta t\, K_1, v_{n+1})$$
$$0 = g(H_n + \Delta t\, K_1, v_{n+1})$$
$$H_{n+1} = H_n + \frac{\Delta t}{2}(K_1 + K_2)$$

- It is not a traditional Runge-Kutta method but a generalized additive Runge-Kutta[2].

1. Arnold, Martin, Karl Strehmel, and Rüdiger Weiner. "Half-explicit Runge-Kutta methods for semi-explicit differential-algebraic equations of index 1." *Numerische Mathematik* 64 (1993): 409-431.
2. Sandu, Adrian, and Michael Günther. "A generalized-structure approach to additive Runge-Kutta methods." *SIAM Journal on Numerical Analysis* 53.1 (2015): 17-42.

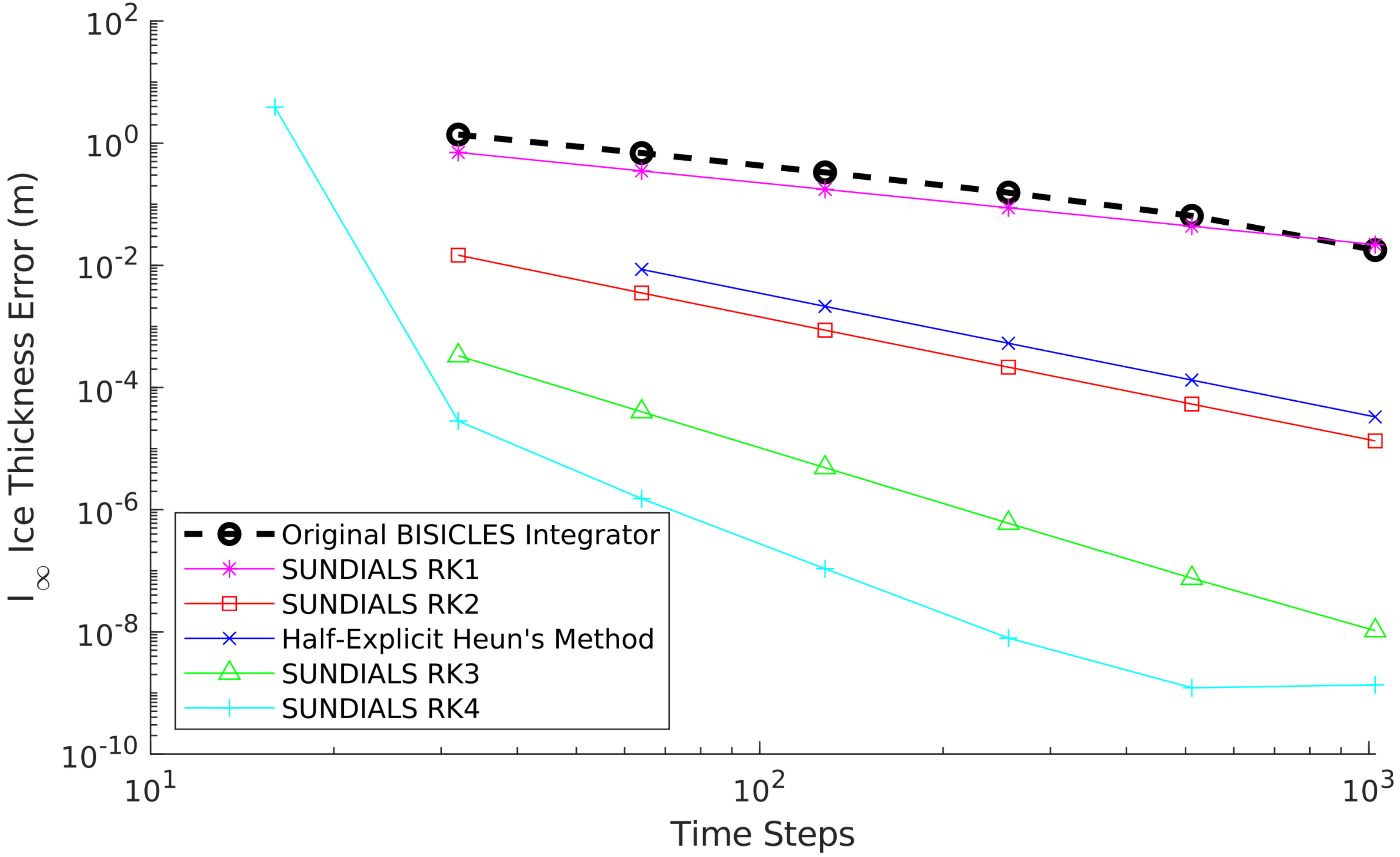
Twisty Steam is a Benchmark Test Problem

- Ice streams are fast-flowing regions within a sheet
- Ice streams account for about 90% of ice mass lost from the Antarctic ice sheet[1]
- We compare the temporal accuracy of
  - The original unsplit Godunov piecewise parabolic method in BISICLES
  - Explicit Runge-Kutta methods from ARKODE of order 1-4
  - The half-explicit Heun's method from the previous slide

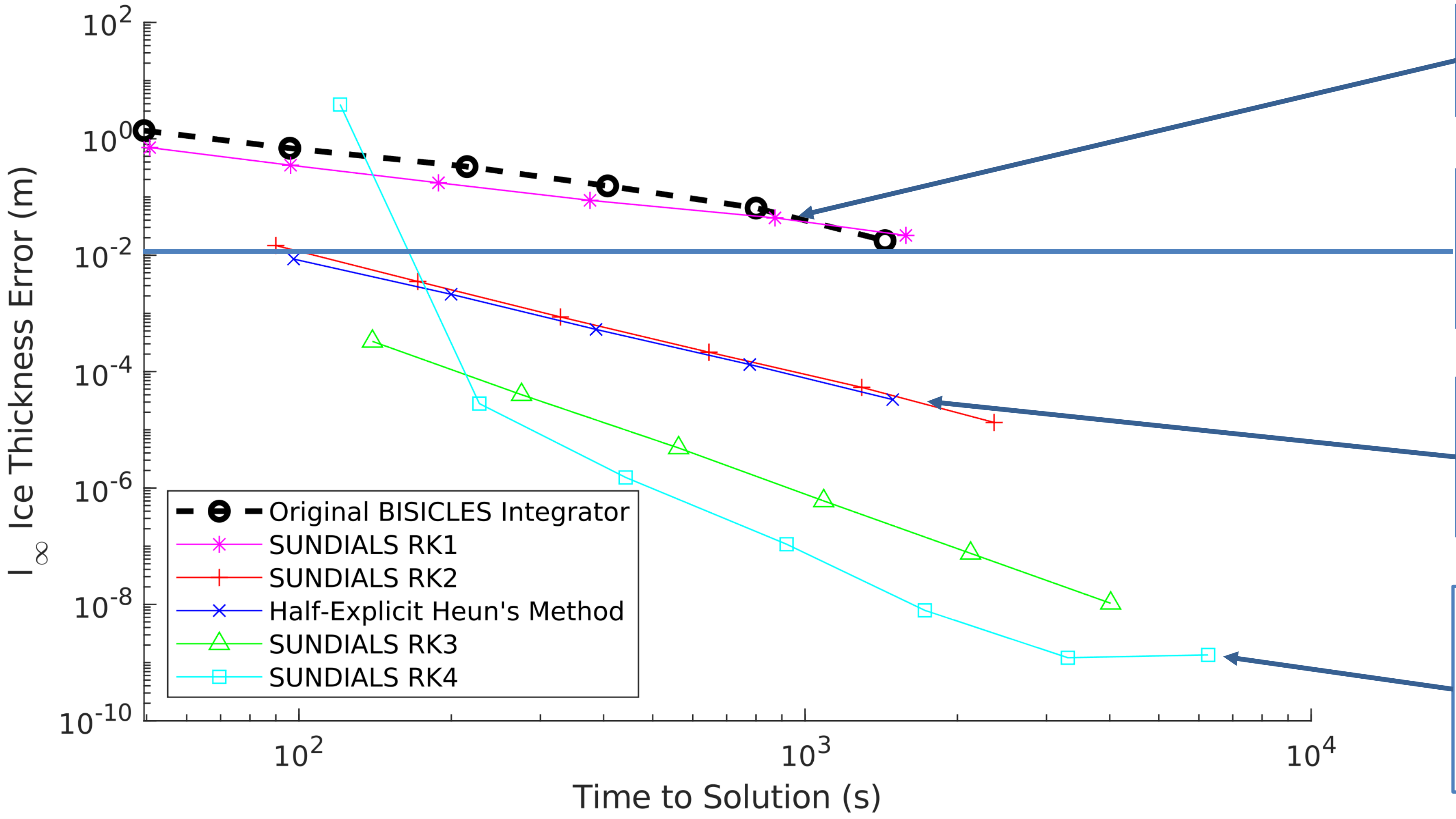


1. https://www.antarcticglaciers.org/glacier-processes/glacier-types/ice-streams/

The Integrators Converge



The New Integrators Are Significantly More Efficient



This overhead of SUNDIALS is negligible

2nd order methods have ~15x speedup at temporal error of 10^{-2}m

The half-explicit Heun’s method has similar performance to RK2

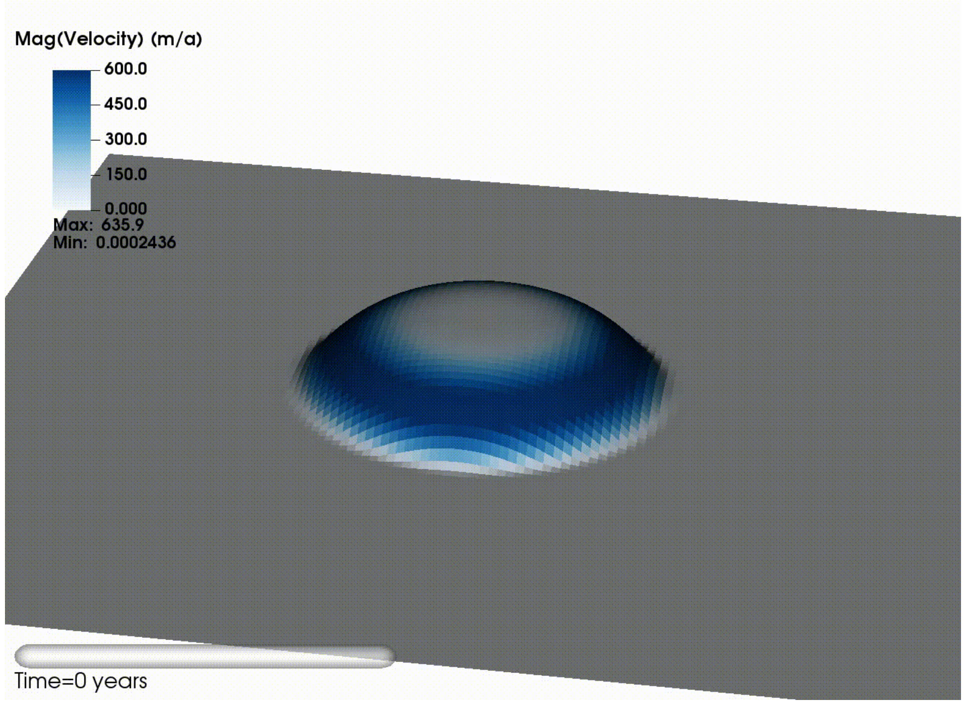
High order methods give orders of magnitude less temporal error for fixed runtime

BISICLES is Often Stability-Limited

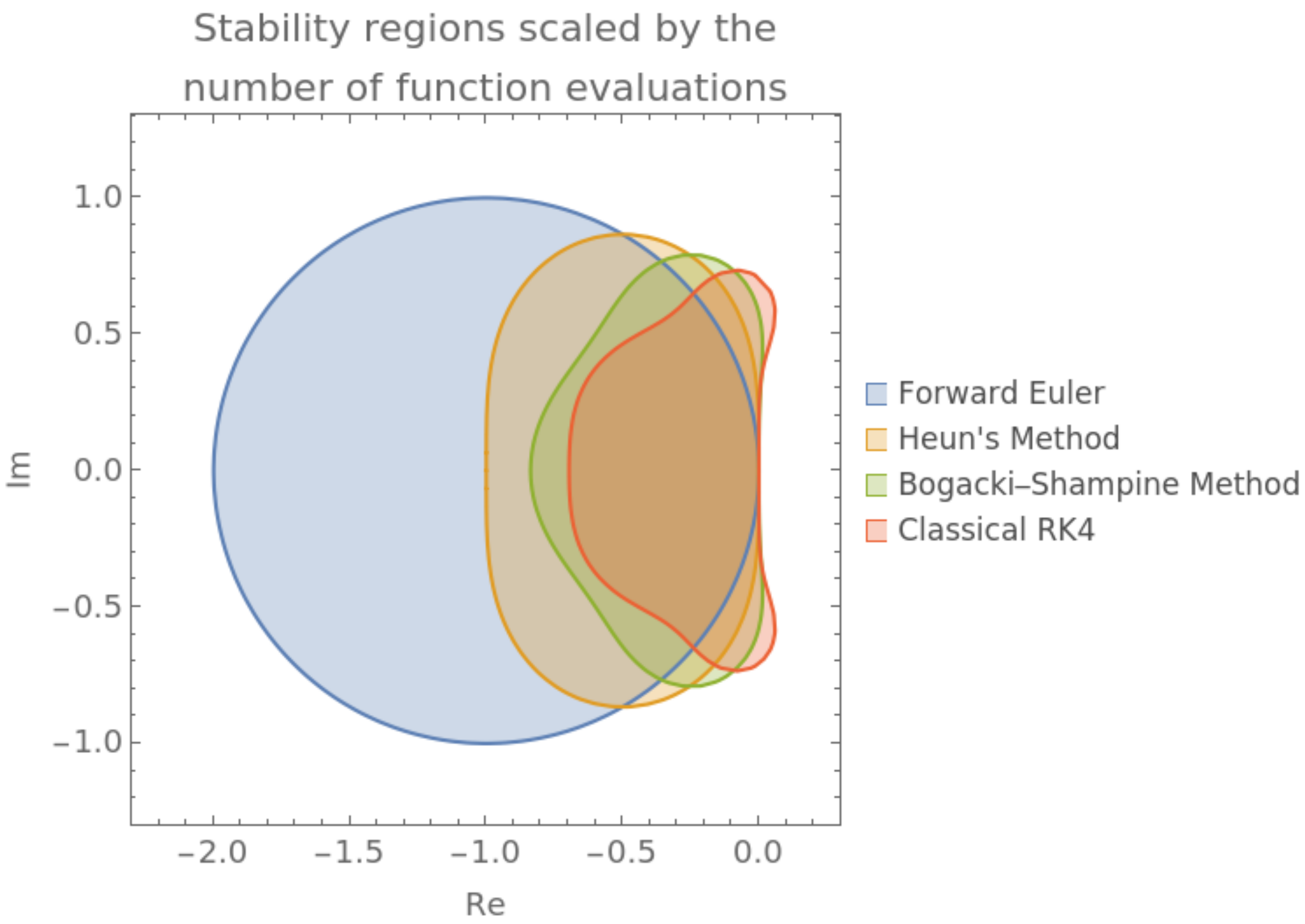
$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}(v_x H) + \frac{\partial}{\partial y}(v_y H)$$
$$\beta^2 v - \nabla \cdot (H\mu(v)\,\nabla v) = -\rho_i g H\,\nabla \cdot s$$

- Despite appearing purely advective, the ice thickness evolves like an advection-diffusion equation due to the advection velocity depending on thickness
- Spatial error often dominates temporal error, even when using the native, first order method
- In this regime, we achieve the best efficiency by taking Δt near the CFL limit
- The following metric is key

$$\frac{\text{max stable } \Delta t}{\text{cost per step}} \approx \frac{\text{extent of linear stability region}}{\text{stages}}$$



High Order is Not Always Advantageous for Linear Stability



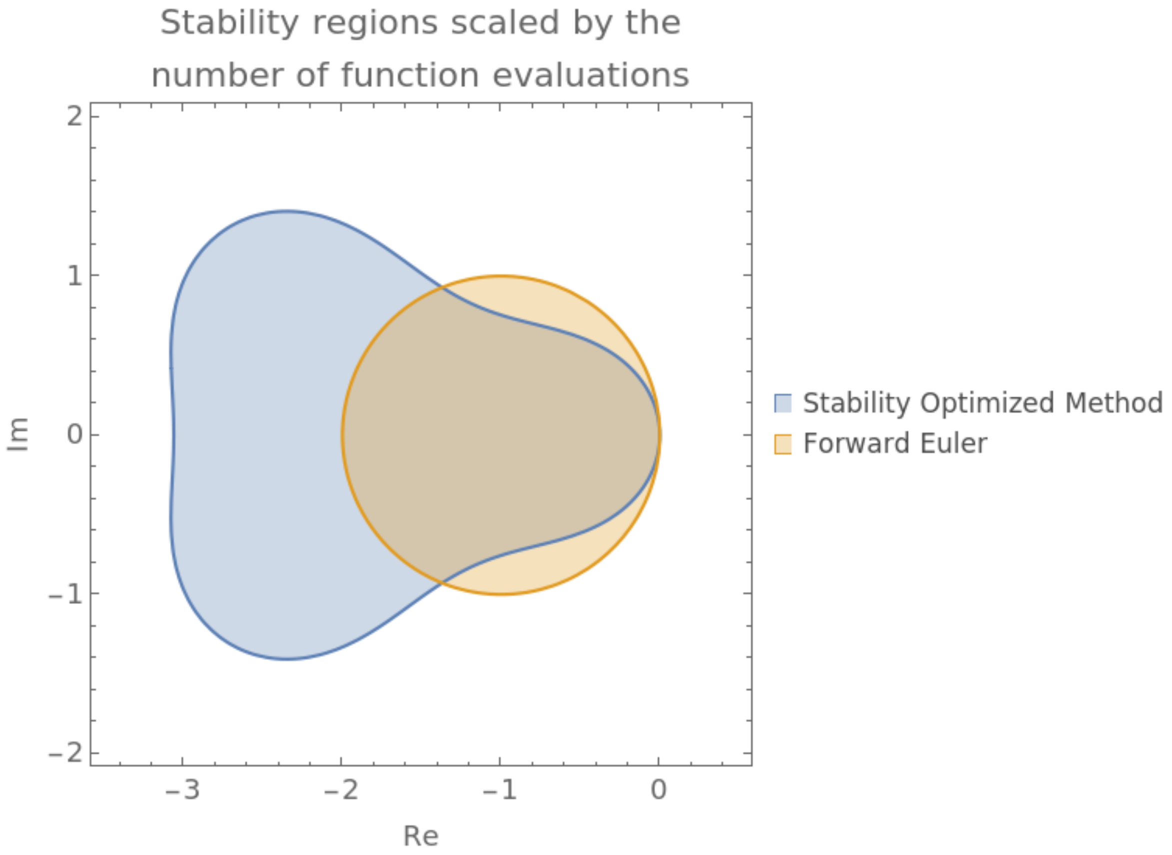
We Can Optimize The Stability with Additional Stages

- We tested a first order method in 3 stages with a large stability region

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & \frac{11}{21} & \frac{1}{7} & 0 \\ \hline & \frac{53}{80} & \frac{3}{40} & \frac{21}{80} \end{array}$$

- For the twisty stream problem, we can take a time step roughly 5x bigger

- The minimum time to a stable solution is reduced by about 35% for the twisty stream problem



Conclusions

- New Runge-Kutta integrators from SUNDIALS facilitate faster and more accurate modeling of ice sheets
- Embedded error estimation offers a simpler and robust alternative to CFL based time step selection
- Chombo N_Vector is now available in Chombo 3.2 patch 8
- Future and ongoing work
  - Testing multirate methods
  - Exploring other stabilized methods
  - Parallel-in-time leveraging SUNDIALS' wrappers for XBraid
  - Exploring more-complex (realistic) ice sheet configurations (grounding-line retreat, realistic Greenland and Antarctic geometries, etc).

Acknowledgements

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computing.llnl.gov/sundials

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