Practical Multirate Time Integration Methods

using General Linear Methods

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VSGC, April 8, 2019

Compute the Future!

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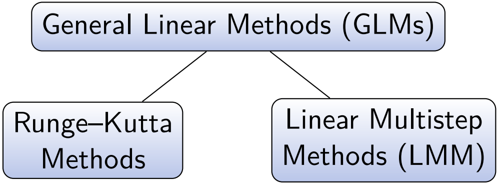
Goal: Efficiently solve systems of ODEs

- Ordinary differential equations (ODEs) are the building blocks for physical models:

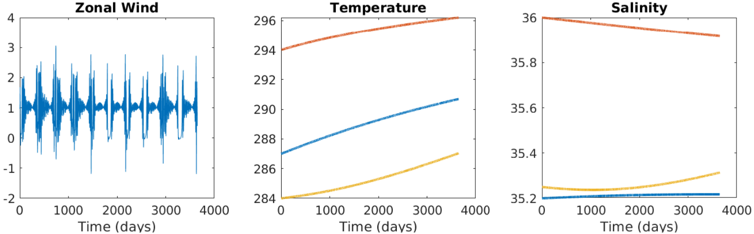
$$y' = f(y), \quad y(t_0) = y_0, \quad f : \mathbb{R}^d \to \mathbb{R}^d.$$

- Some popular numerical methods to solve ODEs are:

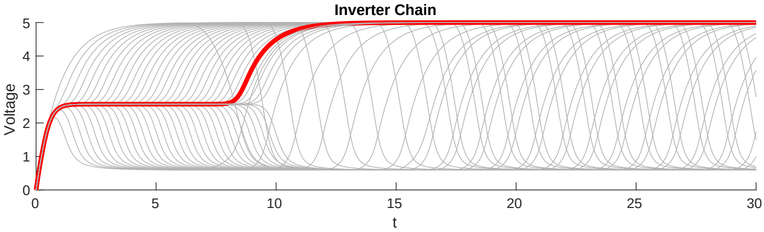


ODEs often exhibit multiscale behavior

- Low order climate model



- Active and latent parts of electric circuit

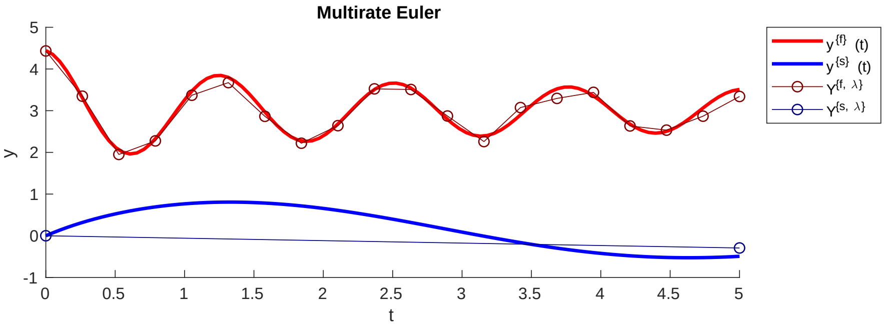


Multirate methods for multiscale problems

- Idea: treat the fast and slow dynamics separately

$$y' = f(y) = f^{\{\mathfrak{f}\}}(y) + f^{\{\mathfrak{s}\}}(y).$$

- Use a timestep H for the slow dynamics
- Use a timestep $h = H/M$ for the fast dynamics



Why multirate general linear methods?

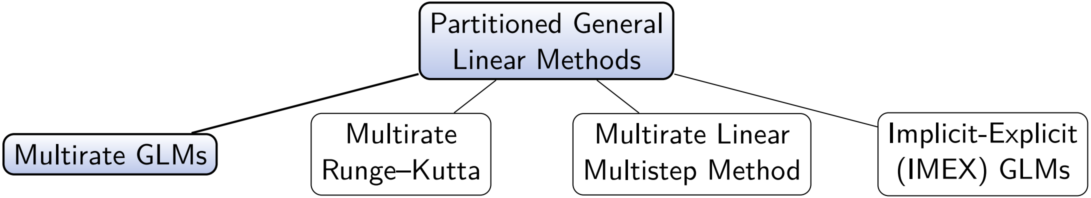
- Existing multirate methods have not gained widespread adoption
- Multirate methods introduce coupling error
- Important tradeoff at high order

Multirate Runge–Kutta	Multirate LMM
Complex order conditions	Simpler order conditions
Challenging (but possible) to have good stability	Dahlquist barriers limit stability

- General linear methods can offer balance of stability and accuracy.

A new framework for partitioned GLMs

- Multirate GLMs do not fit in existing frameworks
- We developed a new partitioned GLM framework which offers extreme flexibility for multimethods



- Order conditions, stability analysis, etc. follow from partitioned GLM theory

One step of a multirate GLM

- Slow internal stages:

$$Y_i^{\{\mathfrak{s}\}[n]} = H \sum_{j=1}^{s^{\{\mathfrak{s}\}}} a_{i,j}^{\{\mathfrak{s},\mathfrak{s}\}} f^{\{\mathfrak{s}\}}\left(Y_j^{\{\mathfrak{s}\}[n]}\right) + h \sum_{\lambda=1}^{M} \sum_{j=1}^{s^{\{\mathfrak{f}\}}} a_{i,j}^{\{\mathfrak{s},\mathfrak{f},\lambda\}} f^{\{\mathfrak{f}\}}\left(Y_j^{\{\mathfrak{f},\lambda\}[n]}\right) + \sum_{j=1}^{r^{\{\mathfrak{s}\}}} u_{i,j}^{\{\mathfrak{s},\mathfrak{s}\}} \xi_j^{\{\mathfrak{s}\}[n-1]} + \sum_{j=1}^{r^{\{\mathfrak{f}\}}} u_{i,j}^{\{\mathfrak{s},\mathfrak{f}\}} \xi_j^{\{\mathfrak{f}\}[n-1]}.$$

- Fast internal stages for $\lambda = 1, \dots, M$:

$$Y_i^{\{\mathfrak{f},\lambda\}[n]} = +H \sum_{j=1}^{s^{\{\mathfrak{s}\}}} a_{i,j}^{\{\mathfrak{f},\mathfrak{s},\lambda\}} f^{\{\mathfrak{s}\}}\left(Y_j^{\{\mathfrak{s}\}[n]}\right) + h \sum_{j=1}^{s^{\{\mathfrak{f}\}}} a_{i,j}^{\{\mathfrak{f},\mathfrak{f}\}} f^{\{\mathfrak{f}\}}\left(Y_j^{\{\mathfrak{f},\lambda\}[n]}\right) + \sum_{j=1}^{r^{\{\mathfrak{s}\}}} u_{i,j}^{\{\mathfrak{f},\mathfrak{s},\lambda\}} \xi_j^{\{\mathfrak{s}\}[n-1]} + \sum_{j=1}^{r^{\{\mathfrak{f}\}}} u_{i,j}^{\{\mathfrak{f},\mathfrak{f}\}} \xi_j^{\{\mathfrak{f}\}[n-1+(\lambda-1)/M]}.$$

- Update to fast external stages:

$$\xi_i^{\{\mathfrak{f}\}[n-1+\lambda/M]} = h \sum_{j=1}^{s^{\{\mathfrak{f}\}}} b_{i,j}^{\{\mathfrak{f},\mathfrak{f}\}} f^{\{\mathfrak{f}\}}\left(Y_j^{\{\mathfrak{f},\lambda\}[n]}\right) + \sum_{j=1}^{r^{\{\mathfrak{f}\}}} v_{i,j}^{\{\mathfrak{f},\mathfrak{f}\}} \xi_j^{\{\mathfrak{f}\}[n-1+(\lambda-1)/M]}.$$

- Update to slow external stages:

$$\xi_i^{\{\mathfrak{s}\}[n]} = H \sum_{j=1}^{s^{\{\mathfrak{s}\}}} b_{i,j}^{\{\mathfrak{s},\mathfrak{s}\}} f^{\{\mathfrak{s}\}}\left(Y_j^{\{\mathfrak{s}\}[n]}\right) + \sum_{j=1}^{r^{\{\mathfrak{s}\}}} v_{i,j}^{\{\mathfrak{s},\mathfrak{s}\}} \xi_i^{\{\mathfrak{s}\}[n-1]}.$$

Order conditions

- Start by picking fast and slow base methods of order p and stage order $q = p$ or $q = p - 1$
- The coupling conditions are

$$\frac{\left(c^{\{\mathfrak{f}\}} + (\lambda - 1)\mathbb{1}_{s^{\{\mathfrak{f}\}}}\right)^k}{M^k k!} - \frac{A^{\{\mathfrak{f},\mathfrak{s},\lambda\}} c^{\{\mathfrak{s}\} \times (k-1)}}{(k-1)!} - U^{\{\mathfrak{f},\mathfrak{s},\lambda\}} q_k^{\{\mathfrak{s},\mathfrak{s}\}} = 0, \quad \lambda = 1, \dots, M,$$

$$\frac{M^k c^{\{\mathfrak{s}\} \times k}}{k!} - \sum_{\lambda=1}^{M} \frac{A^{\{\mathfrak{s},\mathfrak{f},\lambda\}} \left(c^{\{\mathfrak{f}\}} + (\lambda - 1)\mathbb{1}_{s^{\{\mathfrak{f}\}}}\right)^{k-1}}{(k-1)!} - U^{\{\mathfrak{s},\mathfrak{f}\}} q_k^{\{\mathfrak{f},\mathfrak{f}\}} = 0,$$

for $k = 1, \dots, q$.

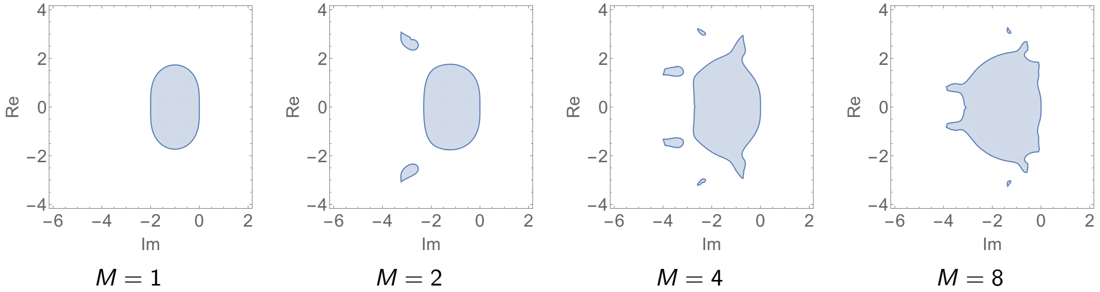
- New explicit methods are derived up to order four

Multirate linear stability analysis

- For which $z = H\lambda$ will a multirate GLM not amplify errors when applied to the following test problem?

$$y' = \underbrace{\frac{M\lambda}{2}y}_{\text{fast}} + \underbrace{\frac{\lambda}{2}y}_{\text{slow}}$$

- Stability regions for a second order multirate GLM

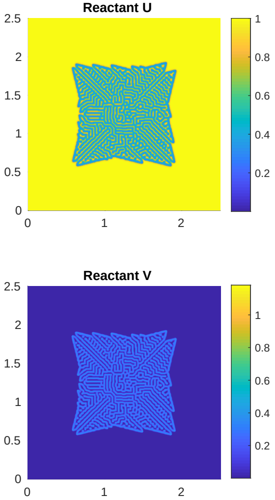


Gray–Scott test problem

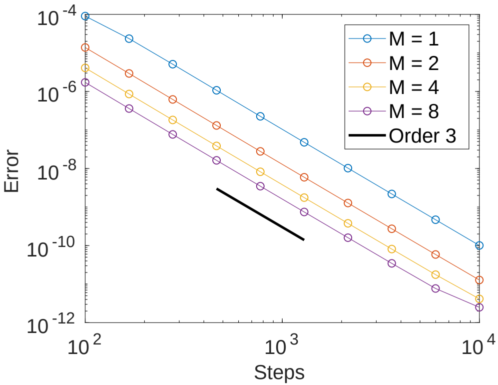
- Gray-Scott reaction diffusion PDE:

$$\underbrace{\begin{bmatrix} u \\ v \end{bmatrix}'}_{y'} = \underbrace{\begin{bmatrix} \varepsilon_u \Delta u \\ \varepsilon_v \Delta v \end{bmatrix}}_{f^{\{\mathfrak{s}\}}(y)} + \underbrace{\begin{bmatrix} -uv^2 + f(1-u) \\ uv^2 - (f+k) \end{bmatrix}}_{f^{\{\mathfrak{f}\}}(y)}.$$

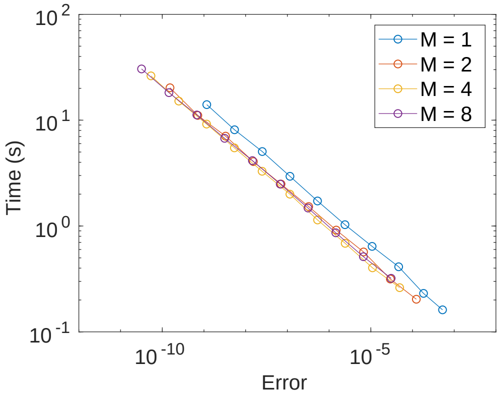
- Periodic boundary conditions on $[0, 2.5]^2$
- Run from $t = 0$ to $t = 100$
- $\varepsilon_u = 2 \times 10^{-6}$, $\varepsilon_v = 10^{-6}$, $f = 0.04$, and $k = 0.06$
- Reaction terms are cheap to compute



Gray–Scott Results



Convergence plot for third order multirate GLM.



Work precision plot for third order multirate GLM.

Conclusion

- Example of a family of multirate GLMs:

$$
\begin{array}{cc|cc}
0 & 0 & 1 & 0 \\
a_{2,1} & 0 & 0 & 1 \\
\hline
\frac{\left(2a_{2,1}-1\right)v_{1,2}+1}{2} & \frac{1-v_{1,2}}{2} & 1-v_{1,2} & v_{1,2} \\
\frac{(2a_{2,1}-1)v_{2,2}}{2} & \frac{4-2a_{2,1}-v_{2,2}}{2} & 1-v_{2,2} & v_{2,2}
\end{array},
$$

$$
A^{\{\mathfrak{f},\mathfrak{s},\lambda\}} = \begin{bmatrix} \frac{(\lambda-1)\left(M-\lambda+(\lambda-1)a_{2,1}+1\right)}{M^2} & 0 \\ \frac{\lambda\left(M-\lambda+\lambda a_{2,1}\right)}{M^2} & 0 \end{bmatrix}, \quad A^{\{\mathfrak{f},\mathfrak{s},\lambda\}} = \begin{bmatrix} \frac{(\lambda-1)\left(M-\lambda+(\lambda-1)a_{2,1}+1\right)}{M^2} & 0 \\ \frac{\lambda\left(M-\lambda+\lambda a_{2,1}\right)}{M^2} & 0 \end{bmatrix},
$$

$$
U^{\{\mathfrak{f},\mathfrak{s},\lambda\}} = \begin{bmatrix} \frac{M^2-(\lambda-1)^2}{M^2} & \frac{(\lambda-1)^2}{M^2} \\ 1-\frac{\lambda^2}{M^2} & \frac{\lambda^2}{M^2} \end{bmatrix}, \quad U^{\{\mathfrak{s},\mathfrak{f}\}} = \begin{bmatrix} 1 & 0 \\ \frac{(1-M)\left(4a_{2,1}-1\right)}{4(M+1)a_{2,1}-4M-1} & \frac{M\left(8a_{2,1}-5\right)}{4(M+1)a_{2,1}-4M-1} \end{bmatrix}.
$$

- We develoedp a new framework for partitioned GLMs
- Utilized framework to construct and analyze multirate GLMs
- New methods show excellent stability
- Numerical experiments confirm theoretical properties and show speedup over singlerate GLMs

Questions

- Slides and report available at `https://steven-roberts.github.io`
- Additional work on practical multirate time integration:
  - Steven Roberts, Arash Sarshar, and Adrian Sandu. "Coupled Multirate Infinitesimal GARK Schemes for Stiff Systems with Multiple Time Scales". In: *arXiv preprint arXiv:1812.00808* (2018)
  - A. Sarshar, S. Roberts, and A. Sandu. "Design of High-Order Decoupled Multirate GARK Schemes". In: *SIAM Journal on Scientific Computing* 41.2 (2019), A816–A847. DOI: 10.1137/18M1182875. eprint: https://doi.org/10.1137/18M1182875. URL: https://doi.org/10.1137/18M1182875